MATHEMATICAL COMMUNICATIONS
Math. Commun. **23**(2018), 279–293

Operational method for the solution of ordinary differential equations using Hermite series

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Received August 31, 2017; accepted January 26, 2018

Abstract. We apply a general algebraic operational method to obtain solutions of ordinary differential equations. The solutions are expressed as series of scaled Hermite polynomials. We present some examples that show that the solutions obtained as truncated Hermite series give acceptable approximations to the exact solutions on intervals larger than the corresponding intervals for the solutions obtained as truncated Taylor series. Our method is algebraic and does not use any integral transforms.

AMS subject classifications: 33C45, 34A25, 44A40

Key words: Operational calculus, ordinary differential equations, series of Hermite polynomials

1. Introduction

Since ordinary differential equations are one of the main tools in mathematical modeling in science and engineering, many different approaches have been used to develop methods to find solutions of such equations, see, for example, [7, 9, 10]. The methods that are used most often are based on the analytical theory of integral transforms, such as the Laplace and the Mellin transform.

In this paper, we present a method based on a new general operational calculus that was introduced in [4]. The operational calculus is an abstract linear algebra theory that can be used to solve many kinds of linear functional equations. Applications of the method can be consulted in [1, 2, 3, 5]. For each kind of functional equation we choose some particular instances of abstract elements in the general theory and obtain what we call a *concrete realization*. We will use a concrete realization that uses normalized and re-scaled Hermite polynomials and gives the solutions of differential equations as convergent series of such Hermite polynomials.

We have two main objectives in this paper. First, we wish to show that the application of the operational method using Hermite polynomials and Hermite series is as simple as using monomials and Taylor series. The second objective is to

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show that, in general, the approximations to the solutions obtained with truncated Hermite series are acceptable over intervals that are larger than the corresponding intervals for truncated Taylor series.

Due to the orthogonality of Hermite polynomials over the real line, a large class of functions can be expressed as convergent series of Hermite polynomials and the estimation of errors in the approximation by truncated Hermite series is straightforward. For the properties of Hermite polynomials see [8].

The paper is organized as follows. In Section 2, we present a brief description of the operational calculus introduced in [4]. In Section 3, we present the concrete realization with Hermite polynomials as the generators of the field of series. Then we show how the method is used to find a particular and general solution of several simple examples, and how the initial conditions can be easily incorporated. We consider examples for which the exact solution can be found, so that we can compute the errors due to the truncation of Hermite and Taylor series. Finally, we compare relative errors obtained with our approximated solutions with those obtained by truncated Taylor series, and we present some graphs of relative errors. Numerical computations were performed using Maple 2015.

2. Operational calculus

In this section, we summarize the general operational calculus introduced in [4]. Let $\{p_k : k \in \mathbb{Z}\}$ be a group with multiplication defined by $p_k p_n = p_{k+n}$, for $k,\ n \in \mathbb{Z}$. Let $\mathcal{F}$ be the set of all formal series of the form

$$a = \sum_{k \in \mathbb{Z}} a_k p_k,$$

where a_k is a complex number for each $k \in \mathbb{Z}$ and either all a_k are equal to zero, or there exists an integer $v(a)$ such that $a_k = 0$ whenever $k < v(a)$ and $a_{v(a)} \neq 0$. In the first case, we write $a = 0$ and define $v(0) = \infty$. Addition in $\mathcal{F}$ and multiplication by complex numbers are defined in the usual way.

We define multiplication in $\mathcal{F}$ by extending multiplication of the group $\{p_k : k \in \mathbb{Z}\}$ as follows. If $a = \sum a_k p_k$ and $b = \sum b_k p_k$ are elements of $\mathcal{F}$, then $ab = c = \sum c_n p_n$, where the coefficients c_n are defined by

$$c_n = \sum_{v(a) \leq k \leq n - v(b)} a_k b_{n-k}.$$

Note that $v(ab) = v(a) + v(b)$ and p_{-n} is the inverse of p_n for $n \in \mathbb{Z}$. This multiplication in $\mathcal{F}$ is associative and commutative and p_0 is its unit element. Define $\mathcal{F}_n = \{a \in \mathcal{F} : v(a) \geq n\}$, $n \in \mathbb{Z}$. It was proved in [4] that $\mathcal{F}$ is a field and that $\mathcal{F}_0$ is a subring of $\mathcal{F}$.

Let x be a complex number. From the definition of multiplication in $\mathcal{F}$ it is easy to verify that

$$(p_0 - x p_1) \sum_{k \geq 0} x^k p_k = p_0. \tag{1}$$

The series $\sum_{k\geq 0} x^k p_k$ is denoted by $e_{x,0}$ and called the *geometric series* associated with x. The element $e_{x,0}$ is the reciprocal of $p_0 - xp_1$. Let

$$e_{x,k} = \frac{p_k}{(p_0 - xp_1)^{k+1}} = p_k(e_{x,0})^{k+1} = \sum_{n\geq k} \binom{n}{k} x^{n-k} p_n, \qquad k \geq 0. \tag{2}$$

Observe that

$$e_{x,k}\left(p_{-k}\,(p_0 - xp_1)^{k+1}\right) = p_0. \tag{3}$$

An important property of the series $e_{x,k}$ is the following multiplication formula. For $x, y \in \mathbb{C}$, such that $x \neq y$, we have

$$p_1 e_{x,m} e_{y,n} = \sum_{k=0}^{m} \frac{\binom{n+k}{k}(-1)^k e_{x,m-k}}{(x-y)^{1+n+k}} + \sum_{j=0}^{n} \frac{\binom{m+j}{j}(-1)^j e_{y,n-j}}{(y-x)^{1+m+j}}, \quad n, m \in \mathbb{N}.$$

A particular case is given by

$$p_1 e_{x,0} e_{y,0} = \frac{e_{x,0} - e_{y,0}}{x - y}. \tag{4}$$

To each nonzero series b there corresponds multiplication map that sends a to ab. This map is clearly linear and invertible. The multiplication map that corresponds to the element p_1 is called the right shift and is denoted by S. Its inverse S^{-1} is called the left shift. Note that $\{S^k : k \in \mathbb{Z}\}$ is a group isomorphic to $\{p_k : k \in \mathbb{Z}\}$.

Denote by P_n the projection on $\langle p_n \rangle$, the subspace generated by p_n. If $a \in \mathcal{F}$ then $P_n a = a_n p_n$. It is easy to see that

$$S^k P_n S^{-k} = P_{n+k}, \qquad k, n \in \mathbb{Z}.$$

We define a linear operator L on $\mathcal{F}$ as follows. $Lp_k = S^{-1}p_k = p_{k-1}$ for $k \neq 0$, and $Lp_0 = 0$. Then, for a in $\mathcal{F}$ we have

$$La = L \sum_{k\geq v(a)} a_k p_k = S^{-1}(a - a_0 p_0) = S^{-1}(I - P_0)a,$$

where I is the identity operator and P_0 is the projection on the subspace $\langle p_0 \rangle$. We call L the *modified left shift*. Note that L is not invertible, since its kernel is the subspace $\langle p_0 \rangle$.

For $k \geq 0$ let $\mathcal{F}_{[0,k]} = \mathrm{Ker}(P_0 + P_1 + \cdots + P_k)$. If $k = 0$, we write $\mathcal{F}_{[0]}$ instead of $\mathcal{F}_{[0,0]}$.

Let

$$w(t) = \prod_{j=0}^{r} (t - x_j)^{m_j+1},$$

where $x_0, x_1, \ldots, x_r$ are distinct complex numbers, $m_0, m_1, \ldots, m_r$ are nonnegative integers, and we set $n + 1 = \sum_j (m_j + 1)$. Then we define the operator

$$w(L) = (L - x_0 I)^{m_0+1}(L - x_1 I)^{m_1+1} \cdots (L - x_r I)^{m_r+1}. \tag{5}$$

Theorem 1 ([4], p. 339)**.** *Let $w(L)$ be as defined in* (5)*. Define*

$$d_w = p_{r+1} e_{x_0,m_0} e_{x_1,m_1} \cdots e_{x_r,m_r},$$

and

$$K_w = \langle e_{x_j,i} : 0 \le j \le r,\ 0 \le i \le m_j \rangle.$$

Then g is in the image of $w(L)$ if and only if $d_w g \in \mathcal{F}_{[0,n]}$, $K_w = \mathrm{Ker}(w(L))$, and for every $g \in \mathrm{Im}(w(L))$ we have $w(L)(d_w g) = g$ and thus

$$\{f \in \mathcal{F} : w(L)f = g\} = \{d_w g + h : h \in K_w\}.$$

3. Concrete realization

Let D be the operator of differentiation with respect to t. With the aim of applying the theory presented in Section 2 to solve differential equations we need to give a concrete definition of the elements p_k such that

$$Dp_k = \begin{cases} p_{k-1}, & k \neq 0, \\ 0, & k = 0. \end{cases} \tag{6}$$

Assuming that we want p_k to be a sequence of polynomials in the variable t, there are infinitely many sequences that satisfy (6). In this paper we will use sequences of normalized Hermite polynomials. The classical Hermite polynomials are defined by the recurrence relation $H_{k+1}(t) = 2tH_k(t) - 2kH_{k-1}(t)$, where $H_0(t) = 1$ and $H_1(t) = 2t$, and they are an orthonormal basis for the space of polynomials with respect to the inner product defined by

$$(H_n(t), H_m(t)) = \frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} w(t) H_n(t) H_m(t) dt,$$

with the weight function $w(t) = \mathrm{e}^{-t^2}$.

Other important properties of the Hermite polynomials are the following:

a) $H_k'(t) = 2kH_{k-1}(t)$,

b) $H_{k+1}(t) = 2tH_k(t) - H_k'(t)$.

By the orthonormality of the Hermite polynomials we can write any polynomial $q(t)$ as a linear combination of Hermite polynomials as follows:

$$q(t) = \sum_{j=0}^{n} (q(t), H_j(t))\ H_j(t).$$

For each nonzero real number a we define the normalized sequence of Hermite polynomials

$$h_{k,a}(t) = \frac{H_k(at)}{a^k 2^k k!}$$

and we set

$$p_k = \begin{cases} h_{k,a}(t), & k \geq 0, \quad a \in \mathbb{R} - \{0\}, \\ \dfrac{t^k}{k!}, & k < 0 \end{cases} \tag{7}$$

where $k!$ is defined for negative k by

$$k! = \frac{(-1)^{-k-1}}{(-k-1)!}, \qquad k < 0.$$

Observe that in (7), the value of a determines a stretching or a contraction of the graphs of Hermite polynomials. It is not difficult to see that $Dp_k = p_{k-1}$, for $k \neq 0$ and $Dp_0 = 0$. The multiplication $p_k p_n = p_{k+n}$ induces a product of Hermite polynomials given by $h_{k,a}(t) * h_{n,a}(t) = h_{k+n,a}(t)$, calling the algebraic convolution product. For example, $h_{1,a}(t) * h_{2,a}(t) = t * (4a^2t^2 - 2)/(8a^2) = h_{3,a}(t) = (8a^3t^3 - 12at)/(a^3 2^3 3!)$. This convolution product does not coincide with the usual convolution of functions defined by integration.

From (1) and (3) we obtain that for $x \in \mathbb{C}$ and $k \in \mathbb{N}$

$$(h_{0,a}(t) - xh_{1,a}(t)) * \sum_{j \geq 0} x^j h_{j,a}(t) = h_{0,a}(t)$$

and

$$\frac{t^{-k}}{(-k)!} * (h_{0,a}(t) - xh_{1,a}(t))_*^{k+1} * \sum_{j \geq k} \binom{j}{k} x^{j-k} h_{j,a}(t) = h_{0,a}(t),$$

where $(\cdot)_*^{k+1} := \underbrace{(\cdot) * (\cdot) * \cdots * (\cdot)}_{(k+1)-times}$.

If x and a are real numbers and $a \neq 0$, the geometric series $e_{x,0}$ in this concrete realization is

$$e_{x,0} = \sum_{k=0}^{\infty} x^k \, h_{k,a}(t).$$

Theorem 2. *For any nonzero real number a we have*

$$e_{x,0} = \exp\left(\frac{-x^2}{4a^2}\right) \, \exp(xt). \tag{8}$$

Proof. The general property $Le_{x,0} = xe_{x,0}$ becomes in this case $D_t e_{x,0} = xe_{x,0}$. Therefore $e_{x,0}$ must be of the form $K(a,x)\exp(xt)$, where $K(a,x)$ is independent of t. Observing that $H_k(0) = 0$ if k is odd, and $H_{2k}(0)/(2k)! = (-1)^k/k!$, and taking $t = 0$, we obtain

$$K(a,x) = \sum_{k=0}^{\infty} x^k h_{k,a}(0) = \sum_{k=0}^{\infty} \frac{x^k H_k(0)}{a^k 2^k k!} = \sum_{k=0}^{\infty} \frac{x^{2k} H_{2k}(0)}{a^{2k} 4^k (2k)!} = \exp\left(\frac{-x^2}{4a^2}\right).$$

Therefore we have

$$e_{x,0} = \exp\left(\frac{-x^2}{4a^2}\right) \, \exp(xt).$$

□

Note that this is a generating function for the polynomials $h_{k,a}(t)$. Taking $x = a$ we see that $K(a, a)$ is constant and takes the value 0.77880078.

The trigonometric functions can also be expressed in terms of the geometric series $e_{x,0}$.

Corollary 1.

$$\frac{e_{i,0} - e_{-i,0}}{2i} = \exp\left(\frac{1}{4a^2}\right) \sin(t),$$

and

$$\frac{e_{i,0} + e_{-i,0}}{2} = \exp\left(\frac{1}{4a^2}\right) \cos(t).$$

A more general result of Theorem 2 can be deduced as follows. From (2) it is easy to see that

$$\frac{D_x^k}{k!} e_{x,0} = e_{x,k},$$

where D_x denotes a derivation with respect to x. On the other hand, from (8) we have that

$$e_{x,0} = \exp\left(\frac{-x^2}{4a^2}\right) \exp(xt).$$

It follows that

$$e_{x,k} = \frac{D_x^k}{k!}\left[\exp\left(\frac{-x^2}{4a^2}\right) \exp(xt)\right].$$

Applying the Leibniz rule we get that

$$e_{x,k} = s(x, t) \exp\left(\frac{-x^2}{4a^2}\right) \exp(xt),$$

where $s(x, t)$ is a polynomial of degree k in both variables.

We have defined a concrete realization of the abstract operational calculus that we can use to solve ordinary differential equations, since the operator L is a differentiation with respect to t. In the next section, we will solve several examples in order to show the effectiveness and simplicity of the method.

4. Implementation of the method

In this section, we solve several linear ordinary differential equations via our method and compare our solutions with the solutions obtained by using the Taylor series.

4.1. Linear ordinary differential equations

Example 1. *Consider a simple equation*

$$\frac{d^2}{dt^2} y(t) + y(t) = t^2. \tag{9}$$

Observe that this equation can be written as

$$(D - iI)(D + iI)y(t) = t^2.$$

It is easy to see that t^2 *can be written as* $p_0/(2a^2) + 2p_2$*. Then the previous equation acquires the form*

$$(L - iI)(L + iI)y = \frac{1}{2a^2}p_0 + 2p_2,$$

where $L = D$ *and* I *is the identity operator. By Theorem 1, a particular solution is given by*

$$y = p_2 e_{i,0} e_{-i,0} \left(\frac{1}{2a^2} p_0 + 2p_2 \right).$$

Using (4) *and simplifying we get*

$$\begin{aligned} y &= \left(\frac{1}{2a^2} p_1 + 2p_3 \right) \left(\frac{e_{i,0} - e_{-i,0}}{2i} \right) \\ &= \left(\frac{1}{2a^2} p_1 + 2p_3 \right) \sum_{k=0}^{\infty} (-1)^k p_{2k+1} \\ &= 2p_2 + \left(\frac{1}{2a^2} - 2 \right) \left(p_0 - \sum_{k=0}^{\infty} (-1)^k p_{2k} \right). \end{aligned}$$

In terms of the concrete realization we obtain ae particular solution

$$y_p(t) = 2h_{2,a}(t) + \left(\frac{1}{2a^2} - 2 \right) \left(h_{0,a}(t) - \sum_{k=0}^{\infty} (-1)^k h_{2k,a}(t) \right). \tag{10}$$

Remark 1. *From Corollary 1 it is easy to verify that*

$$\sum_{k=0}^{\infty} (-1)^k p_{2k+1} = \sum_{k=0}^{\infty} (-1)^k h_{2k+1,a}(t) = \exp\left(\frac{1}{4a^2} \right) \sin(t),$$

and

$$\sum_{k=0}^{\infty} (-1)^k p_{2k} = \sum_{k=0}^{\infty} (-1)^k h_{2k,a}(t) = \exp\left(\frac{1}{4a^2} \right) \cos(t).$$

Therefore (10) *is a linear combination of the cosine function and a quadratic polynomial.*

Example 2. *Consider the equation*

$$\frac{d^2}{dt^2} y(t) - 2\frac{d}{dt} y(t) + y(t) = \exp(t). \tag{11}$$

By (8) *we have* $e_{x,0} = K_{a,x} \exp(xt)$*, with* $K_{a,x} = \exp\left(-x^2/(4a^2)\right)$*. Then the previous equation can be rewritten as*

$$(L - I)^2 y = M e_{1,0},$$

where

$$M = \exp\left(\frac{1}{4a^2}\right).$$

By Theorem 1, a particular solution is given by $y = M\, p_1\, e_{1,1}\, e_{1,0} = M\, e_{1,2}$*, and in terms of the concrete realization we get*

$$y_p(t) = M \sum_{j=2}^{\infty} \binom{j}{2} h_{j,a}(t). \tag{12}$$

Example 3. *Consider the equation*

$$\frac{d}{dt} y(t) - y(t) = \exp(-t^2). \tag{13}$$

The function $\exp(-t^2)$ *is written as the Hermite series [6]*

$$\exp(-t^2) = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(j+1/2)}{\sqrt{2\pi}} \frac{H_{2j}(t)}{2^j (2j)!} = \sum_{j=0}^{\infty} \frac{(-2)^j \Gamma(j+1/2)}{\sqrt{2\pi}}\, h_{2j,1}(t).$$

Now, each polynomial $h_{2j,1}(t)$ *can be expressed as a linear combination of the polynomials* $h_{2k,a}(t)$*, that is,*

$$h_{2j,1}(t) = \sum_{k=0}^{j} \frac{1}{(j-k)!} \left(\frac{1-a^2}{4a^2}\right)^{j-k} h_{2k,a}(t).$$

Hence

$$\exp(-t^2) = \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{(-2)^j \Gamma(j+1/2)}{\sqrt{2\pi}} \left(\frac{1-a^2}{4a^2}\right)^{j-k} \frac{h_{2k,a}(t)}{(j-k)!} = \sum_{k=0}^{\infty} c_{k,a} h_{2k,a}(t),$$

where

$$c_{k,a} = \sum_{n=0}^{\infty} \frac{(-2)^{k+n} \Gamma(k+n+1/2)}{\sqrt{2\pi}} \frac{1}{n!} \left(\frac{1-a^2}{4a^2}\right)^{n}.$$

Thus equation (13) *can be rewritten as*

$$(L - I)y = \sum_{k=0}^{\infty} c_{k,a} p_{2k}.$$

By Theorem 1, a particular solution is given by

$$\begin{aligned} y &= p_1 e_{1,0} \sum_{k=0}^{\infty} c_{k,a} p_{2k} \\ &= \left(\sum_{j=1}^{\infty} p_j\right) \left(\sum_{k=0}^{\infty} c_{k,a} p_{2k}\right). \end{aligned}$$

In terms of the concrete realization

$$y_p(t) = \sum_{k=0}^{\infty} c_{k,a} \sum_{j=1}^{\infty} h_{2k+j,a}(t). \tag{14}$$

In the next section, we perform several numerical computations of our solutions and introduce the initial conditions. In each case, we determinate values of the parameter a for which our solution gives a good approximation of the exact solution on an interval larger than the corresponding interval for the approximation of the solution using truncated Taylor series.

4.2. Initial conditions and numerical computations

In this section, we show how to introduce the initial conditions in our method. For this, we use the characterization of the null space K_w given in Theorem 1. In our computations we use the first 60 terms of the Hermite series.

Example 4. *Consider Example 1 with initial conditions* $y(0) = 1$ *and* $y'(0) = 0$. *By Theorem 1 we have*

$$\mathrm{Ker}((D - iI)(D + iI)) = \langle e_{i,0}, e_{-i,0} \rangle .$$

Thus, a general solution is given by

$$y(t) = c_1 e_{i,0} + c_2 e_{-i,0} + y_p(t),$$

where $y_p(t)$ *is a particular solution obtained in* (10). *Using the initial conditions and* (8) *we obtain a linear system of equations*

$$\begin{aligned} c_1 \exp\left(\frac{1}{4a^2}\right) + c_2 \exp\left(\frac{1}{4a^2}\right) + y_p(0) = 1 \\ ic_1 \exp\left(\frac{1}{4a^2}\right) - ic_2 \exp\left(\frac{1}{4a^2}\right) + y_p'(0) = 0. \end{aligned}$$

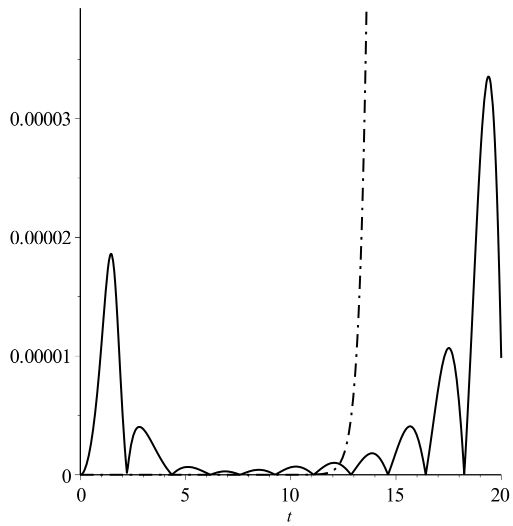
We solve the system for the constants c_1, c_2, *that depend on the parameter* a. *The solution of* (9) *with initial conditions* $y(0) = 1$ *and* $y'(0) = 0$ *is given by*

$$y_M(t) = 3\cos(t) + t^2 - 2,$$

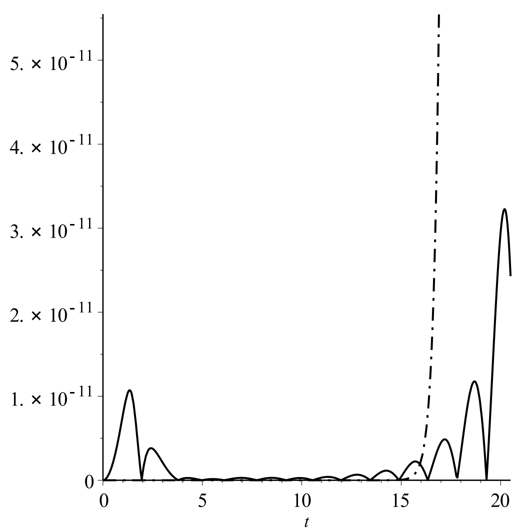
which coincides with Remark 1. Let $y_T(t)$ *be the Taylor series (60 terms, 32 digits) around the origin of* $y_M(t)$. *We observe that the relative error between* $y_M(t)$ *and* $y_T(t)$ *remains below* 1×10^{-3} *when* $0 \le t \le 22.8$. *In Table 1, we include the relative error between* $y_M(t)$ *and our solution truncated at 60 terms for several values of* a. *In Figure 1, we show the two solutions in different cases, for truncated series with 40 and 60 terms.*

Values of a	Relative error is less than 1×10^{-3} when
$a = 1$	$0 \leq t \leq 25$
$a = 0.7$	$0 \leq t \leq 25.1$
$a = 0.5$	$0 \leq t \leq 26.7$
$a = 0.2$	$0 \leq t \leq 37.5$

Table 1: *Values of a and t for which the relative error is kept of order less than 1×10^{-3}*



(a) 40 *terms*



(b) 60 *terms*

Figure 1: *Comparison of relative errors of the Taylor series (−·−) and our solution (—), $a = 0.2$, for 40 and 60 terms in the series*

Example 5. *Consider Example 2 with initial conditions $y(0) = 0$ and $y'(0) = 1$. By Theorem 1 we have that*

$$\mathrm{Ker}((D-I)^2) = \langle e_{1,0}, e_{1,1} \rangle .$$

Therefore a general solution is given by

$$y(t) = c_1 e_{1,0} + c_2 e_{1,1} + y_p(t),$$

where $y_p(t)$ is a particular solution obtained in (12). Using the initial conditions we obtain a linear system of equations

$$c_1 \sum_{j=0}^{\infty} h_{j,a}(0) + c_2 \sum_{j=1}^{\infty} j h_{j,a}(0) + y_p(0) = 1$$
$$c_1 \sum_{j=0}^{\infty} h_{j,a}(0) + c_2 \sum_{j=0}^{\infty} (j+1) h_{j,a}(0) + y_p'(0) = 0.$$

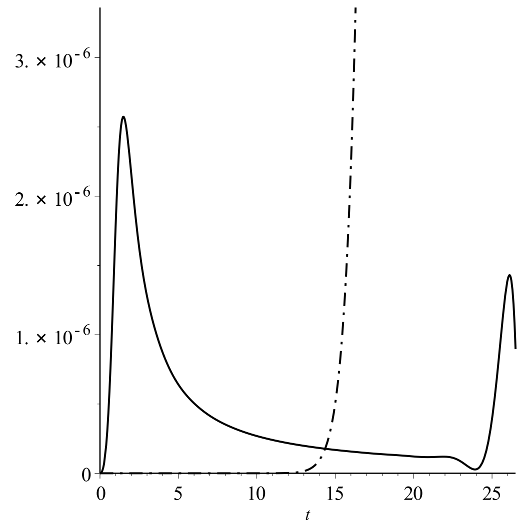
We solve the system for c_1, c_2, that depend on the parameter a. The solution of (11) with initial conditions $y(0) = 1$ and $y'(0) = 0$ is given by

$$y_M(t) = \mathrm{e}^t - t\mathrm{e}^t + \frac{1}{2} t^2 \mathrm{e}^t .$$

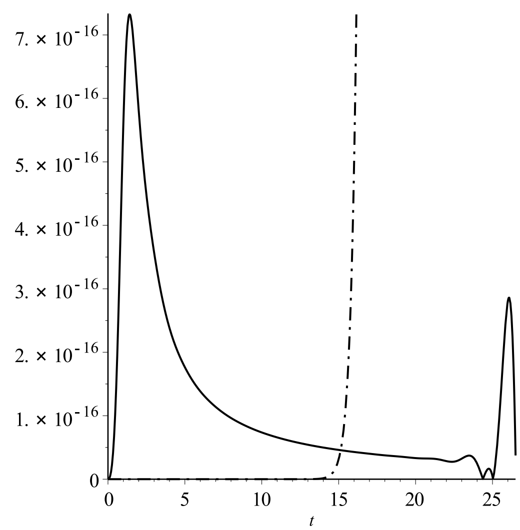
Let $y_T(t)$ be the Taylor series (60 terms, 32 digits) around the origin of $y_M(t)$. We observe that the relative error between $y_M(t)$ and $y_T(t)$ remains below 1×10^{-3} when $0 \le t \le 38.4$. In Table 1, we include the relative error between $y_M(t)$ and our solution truncated at 60 terms for several values of a. In Figure 2, we show the two solutions in different cases, using truncated series with 40 and 60 terms.

Values of a	Relative error is less than 1×10^{-3} when
$a = 1$	$0 \le t \le 39.9$
$a = 0.7$	$0 \le t \le 40.6$
$a = 0.5$	$0 \le t \le 41.9$
$a = 0.31$	$0 \le t \le 46.1$

Table 2: *Values of a and t for which the error is kept of order less than 1×10^{-3}*



(a) *40 terms*



(b) *60 terms*

Figure 2: *Comparison of relative errors of the Taylor series (−·−) and our solution (—), $a = 0.31$, for 40 and 60 terms in the series*

Example 6. *Consider Example 3 with initial condition $y(0) = 1$. By Theorem 1 we have that*

$$\mathrm{Ker}(D - I) = \langle e_{1,0} \rangle .$$

So, a general solution is given by

$$y(t) = c_1 \exp\left(\frac{-1}{4a^2}\right) \exp(t) + y_p(t),$$

where $y_p(t)$ is a particular solution obtained in (14). *Using the initial conditions and* (8) *we obtain the linear equation*

$$c_1 \exp\left(\frac{-1}{4a^2}\right) + y_p(0) = 1,$$

whose solution is

$$c_1 = (1 - y_p(0)) \exp\left(\frac{1}{4a^2}\right).$$

Observe that the constant c_1 depends on the parameter a. The solution of (11) *with the initial condition $y(0) = 1$ obtained with Maple 2015 is given by*

$$y_M(t) = \left(1/2\,\sqrt{\pi}\mathrm{e}^{1/4}\mathrm{erf}\,(t + 1/2) + 1 - 1/2\,\sqrt{\pi}\mathrm{e}^{1/4}\mathrm{erf}\,(1/2)\right) \mathrm{e}^t,$$

where $\mathrm{erf}(t)$ *is the error function defined by*

$$\mathrm{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t \mathrm{e}^{-x^2} dx.$$

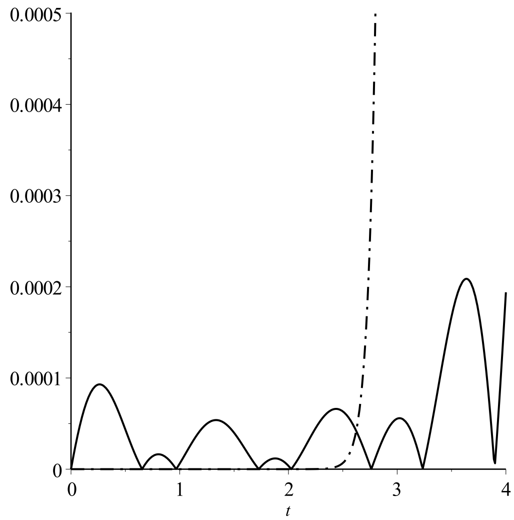
Let $y_T(t)$ be the Taylor series (60 terms, 32 digits) around the origin of $y_M(t)$. We observe that the relative error between $y_M(t)$ and $y_T(t)$ remains below 1×10^{-3} when $0 \le t \le 3.31$. In Table 6, we include the relative error between $y_M(t)$ and our solution truncated at 60 terms for several values of a. In Figure 3, we show the two solutions in different cases where in each case we change the number of terms in the series (40 and 60).

Values of a	Absolute error is less than 1×10^{-3} when
$a = 1$	$0 \le t \le 5.3$
$a = 0.9$	$0 \le t \le 5.8$
$a = 0.8$	$0 \le t \le 5.5$

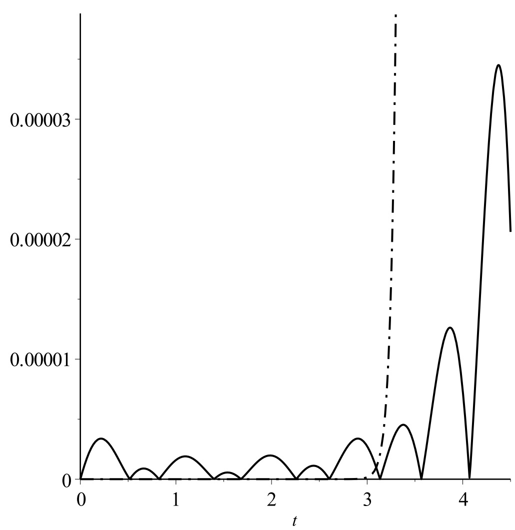
Table 3: *Values of a and t for which the error is kept of order less than 1×10^{-3}*

5. Conclusions

From the examples we can see that the operational method is easy to apply, it uses simple algebraic operations with series, and does not require integration or the use of integral transforms. Note that the method can be applied to any non-homogeneous equations whose right-hand side (forcing function) can be expressed as a Hermite series. In the case of simple equations, such as that in Example 4.1, we can identify the exact solution, represented as a Hermite series.



(a) *40 terms*



(b) *60 terms*

Figure 3: *Comparison of relative errors of the Taylor series (−·−) and our solution (—), $a = 0.9$, for 40 and 60 terms in the series*

Regarding numerical approximations with truncated Hermite series, because of the orthogonality properties, truncated Hermite series have better approximation properties than truncated Taylor series over relatively large intervals. The numerical results reported above confirm this fact. Choosing appropriate values for the scaling parameter a we obtain approximations that yield a small relative error over intervals larger than the corresponding intervals for the truncated Taylor series. The numerical results also show the known fact that the Taylor approximation is better in a small neighborhood of zero. In Examples 4 and 5, it is clear that our solution improves the solution in Taylor series. In Example 6, the improvement is minor, because the values of the solution are small, and our solution only improves the Taylor approximation in a slightly larger interval. It is worth mentioning that, in our computations, handling the parameter a does not cause any difficulties. The incorporation of initial conditions is simple and straightforward.

Our examples were chosen in such way that the "exact" errors could be computed. We can infer that the solutions, expressed as Hermite series, of problems for which the exact solution is not known can be used with reasonable confidence.

Using the ideas introduced in [4], our algebraic method can be generalized to solve other types of differential equations, such as linear equations with variable coefficients, some nonlinear equations, and partial differential equations.

Acknowledgments

The research reported in this paper was partially supported by grant SEP-CONACYT 220603. The first author was supported by SEP through the PRODEP under the project UAM-PTC-630. The third author was supported by Portuguese National Funds through the FCT Foundation for Science and Technology under the project PEst-UID/EEA/00066/2013.

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